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Nondominated equilibrium solutions of a multiobjective two-person nonzero-sum game in extensive form and corresponding mathematical programming problem

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Abstract In most of studies on multiobjective noncooperative games, games are represented in normal form and a solution concept of Pareto equilibrium solutions which is an extension of Nash equilibrium solutions has been focused on. However, for analyzing economic situations and modeling real world applications, we often see cases where the extensive form representation of games is more appropriate than the normal form representation. In this paper, in a multiobjective two-person nonzero-sum game in extensive form, we employ the sequence form of strategy representation to define a nondominated equilibrium solution which is an extension of a Pareto equilibrium solution, and provide a necessary and sufficient condition that a pair of realization plans, which are strategies of players in sequence form, is a nondominated equilibrium solution. Using the necessary and sufficient condition, we formulate a mathematical programming problem yielding nondominated equilibrium solutions. Finally, giving a numerical example, we demonstrate that nondominated equilibrium solutions can be obtained by solving the formulated mathematical programming problem.

Keywords Nondominated equilibrium solution · Multiobjective two-person nonzero-sum game in extensive form · Mathematical programming problem

1. Introduction

A solution concept of Pareto equilibrium solutions which is an extension of Nash equilibrium solutions has been playing a central role in development of theories and methodologies on multiobjective noncooperative games (Shapley 1959; Zeleny 1975; Corley 1985; Borm et al. 1988, 2003; Wierzbicki 1990; Charnes et al. 1990; Zhao 1991; Wang 1993; Nishizaki and Sakawa 1995, 2000; Voorneveld et al. 1999, 2000). In multiobjective optimization, the concept of Pareto optimal solutions is extended to nondominated solutions by using domination

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cones (Yu 1974, Tamura and Miura 1979). Applying this extension, Nishizaki and Notsu (2007) consider nondominated equilibrium solutions in multiobjective two-person nonzero-sum game in normal form. Most of studies on multiobjective noncooperative games have employed mainly the normal form representation, and examine existence of the solutions and computational methods for obtaining them. However, for analyzing economic situations and modeling real world applications, there are a large number of examples in which the extensive form representation of games is more appropriate than the normal form representation. Namely, through game representation of extensive form in which series of moves and choices of players are described by a tree model, many economic situations with multiple moves of players are directly and appropriately modeled.

By transforming a game in extensive form into a game in normal form, Nash equilibrium solutions can be computed, but unfortunately the number of pure strategies increases exponentially with a size of game. Although alternatively an expected payoff can be expressed as a function of behavior strategies, it becomes a high-degree nonlinear function when the number of levels of the game tree is large. Then it becomes difficult to compute Nash equilibrium solutions. Von Stengel (1996) and Koller et al. (1996) define a sequence which is a path from the root of a game tree to a node, and formulate an expected payoff by giving a probability distribution to a set of sequences which is called a realization plan. In this formulation, the expected payoff is linear even if the game tree becomes large and multistage, and the number of sequences increases linearly with a size of game.

Extension of games in extensive form under a multiobjective environment is made by Krieger (2003), and existence of Pareto equilibrium solutions is considered. In this paper, employing the sequence form, we deal with a multiobjective two-person nonzero-sum game in extensive form, and define a nondominated equilibrium solution based on domination cones. After giving a necessary and sufficient condition for a pair of realization plans to be a nondominated equilibrium solution, we formulate a mathematical programming problem yielding nondominated equilibrium solutions by using the necessary and sufficient condition. Finally, a numerical example is given to demonstrate that nondominated equilibrium solutions can be obtained by solving the formulated mathematical programming problem.

2. Nondominated equilibrium solutions of a multiobjective two-person nonzero-sum game in extensive form

2.1 A multiobjective two-person nonzero-sum game and sequences in the extensive game

A game in extensive form is characterized by a game tree, players, information sets, chance moves, and payoff functions. A game tree is represented by a graph with nodes including the root which is an initial node and directed edges. Especially, a terminal node is called a leaf, and at each of leaves a vector of payoffs is assigned to each player in multiobjective games. An example of a multiobjective two-person nonzero-sum game in extensive form is given in Fig. 1, where $n_i, i = 1, \dots, 31$ denote nodes; $m_i, l_i, i = 1, \dots, 6$ denote choices of player 1; $c_i, d_i, i = 1, 2$ denote choices of player 2; and $p_i, i = 1, 2$ denote probabilities of the chance move.

There are two representations of strategies in an extensive form game: behavior strategies and mixed strategies in the corresponding normal form game. An expected payoff as a function of behavior strategies becomes a high-degree nonlinear function when the number of levels of the game tree is large. When an extensive form game is transformed into a normal form game, the number of pure strategies increases exponentially with a size of game. On the

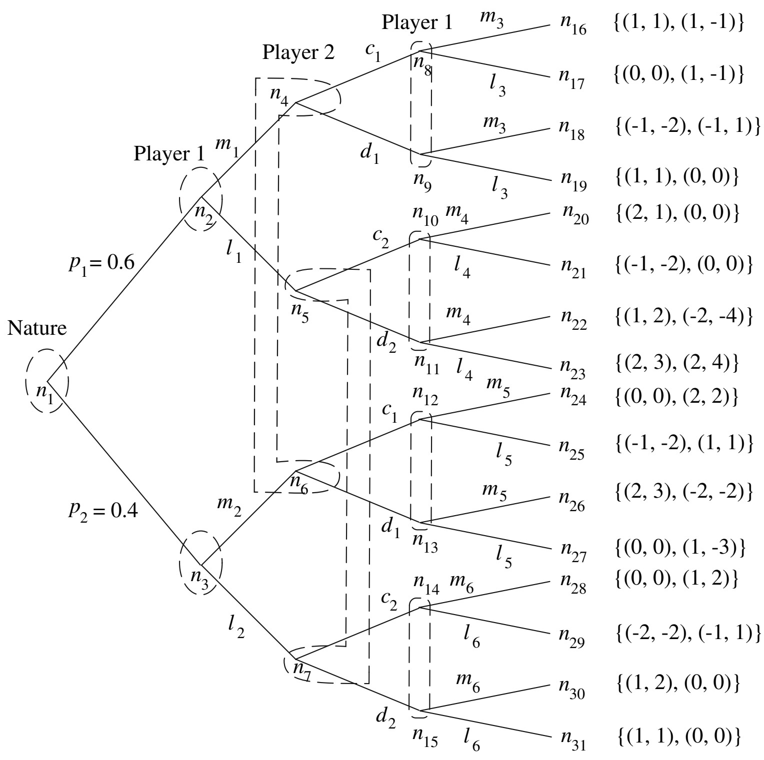

Fig. 1 A game tree of a multiobjective two-person nonzero-sum game

assumption of perfect recall of players, von Stengel (1996) and Koller et al. (1996) propose a game representation of the sequence form which does not cause the mentioned above difficulties. Namely, the expected payoff as a function of realization plans is linear even if the game tree becomes multistage, and the number of sequences increases linearly with a size of game. Because the exponential increase of the number of pure strategies in the normal form game results from extreme increase of the number of pure strategies such that players' choices are not consistent with behaviors of perfect recall, it can be interpreted that a set of pure strategies in sequence form corresponds to that of normal form excluding not perfect recall pure strategies.

A series of nodes and edges from the root to some node is called a path, and a sequence is defined by a set of labels of edges on the path to the node. For example, for node n_{12} of the game tree depicted in Fig. 1, a sequence of player 1 is m_2, that of player 2 is c_1, and that of chance player is p_2. For node n_{25} which is a leaf, a sequence of player 1 is $m_2 l_5$, and those of player 2 and chance player are the same as the sequences for node n_{12}.

Let L be a set of leaves. Payoff functions in extensive form are defined on the set L, and a vector of payoffs is assigned to each of the players at any leaf $l \in L$; let $H_1 : L \to \mathbb{R}^{r_1}$ be the payoff function of player 1, and let $H_2 : L \to \mathbb{R}^{r_2}$ be that of player 2, where r_1 and r_2 are the numbers of payoffs (objectives) of players 1 and 2, respectively. In contrast, payoff functions in sequence form are defined on a set of sequences. Let S_0, S_1, and S_2 be the sets of sequences of chance player, player 1, and player 2, respectively, and let $|S_0|$, $|S_1|$, and

$|S_2|$ be the numbers of sequences of chance player, player 1, and player 2, respectively. Let $S = S_0 \times S_1 \times S_2$ be the space of sequences of all the players.

A payoff function of player 1 in sequence form is defined as $G_1 : S \to \mathbb{R}^{r_1}$, and if a sequence $\mathbf{s} = (s_0, s_1, s_2) \in S$ is specified at a leaf $l \in L$, the payoff function is $G_1(\mathbf{s}) = H_1(l)$ and otherwise it is $G_1(\mathbf{s}) = \mathbf{0}$. A payoff function of player 2 $G_2 : S \to \mathbb{R}^{r_2}$ is also defined similarly. For example, for node n_{12} of the game tree depicted in Fig. 1, a sequence vector is $\mathbf{s}^{12} = (p_2, m_2, c_1)$, and payoffs of players 1 and 2 are $G_1(\mathbf{s}^{12}) = (0, 0)$, $G_2(\mathbf{s}^{12}) = (0, 0)$, respectively. For node n_{25} which is a leaf, a sequence vector is $\mathbf{s}^{25} = (p_2, m_2 l_5, c_1)$, and payoffs of players 1 and 2 are $G_1(\mathbf{s}^{25}) = (-1, -2)$, $G_2(\mathbf{s}^{25}) = (1, 1)$, respectively.

A set of all nodes in a game tree is divided into information sets. Let U_1 and U_2 be the sets of information sets of players 1 and 2, respectively, and let $|U_1|$ and $|U_2|$ be the numbers of the information sets of players 1 and 2, respectively. Each information set u exactly belongs to one player i. All nodes in an information set u have the same choices, and the set of choices at u is denoted by C_u. Let $|C_u|$ be the number of choices at u.

Because it is assumed that perfect recall holds for all the players in a sequence form game, all nodes in an information set u have the same sequence. Let the sequence be denoted by σ_u, and it leads the information set u. A choice $c \in C_u$ in u extends the sequence σ_u, and the extended sequence is expressed by $\sigma_u c$, i.e.,

$$\sigma_u c = \sigma_u \cup \{c\}, \quad c \in C_u. \tag{1}$$

With this notation, a set of sequences of player i can be represented by $S_i = \{\emptyset\} \cup \{\sigma_u c \mid u \in U_i, c \in C_u\}$.

In sequence form, a strategy is represented by giving a probability distribution to a set of sequences, and it is called a realization plan. A realization plan $\mathbf{x} \in \mathbb{R}^{|S_1|}$ of player 1 is subject to the following constraints.

$$x(\emptyset) = 1 \tag{2a}$$

$$-x(\sigma_{u_1}) + \sum_{c_1 \in C_{u_1}} x(\sigma_{u_1} c_1) = 0, \quad u_1 \in U_1 \tag{2b}$$

$$x(s_1) \geqq 0, \quad s_1 \in S_1. \tag{2c}$$

Player 2's realization plan $\mathbf{y} \in \mathbb{R}^{|S_2|}$ is also subject to the following constraints.

$$y(\emptyset) = 1 \tag{3a}$$

$$-y(\sigma_{u_2}) + \sum_{c_2 \in C_{u_2}} y(\sigma_{u_2} c_2) = 0, \quad u_2 \in U_2 \tag{3b}$$

$$y(s_2) \geqq 0, \quad s_2 \in S_2. \tag{3c}$$

By using the $(1 + |U_1|) \times |S_1|$ constraint matrix E^1 and the $(1 + |U_2|) \times |S_2|$ constraint matrix E^2, the above constraints (2) and (3) can be simply expressed by

$$E^1 \mathbf{x} = \mathbf{e}^1 \tag{4}$$

$$E^2 \mathbf{y} = \mathbf{e}^2, \tag{5}$$

respectively, where $\mathbf{e}^1$ and $\mathbf{e}^2$ are the $(1 + |U_1|)$- and $(1 + |U_2|)$-dimensional vectors such that the first element is 1 and the other elements are all 0, i.e., $(1, 0, \ldots, 0)^T$. The superscript T means the transposition of a vector or a matrix. Then, the sets X and Y of realization plans of players 1 and 2 are defined by

$$X = \left\{ \mathbf{x} \in \mathbb{R}^{|S_1|} \mid E^1\mathbf{x} = \mathbf{e}^1,\ \mathbf{x} \geqq \mathbf{0} \right\} \tag{6}$$

$$Y = \left\{ \mathbf{y} \in \mathbb{R}^{|S_2|} \mid E^2\mathbf{y} = \mathbf{e}^2,\ \mathbf{y} \geqq \mathbf{0} \right\}, \tag{7}$$

respectively.

Let $\mathbf{p} = (p_1, \ldots, p_{|S_0|})$ be a realization plan of chance player. When players 1 and 2 choose sequences s_1 and s_2, respectively, the expected payoffs of them are

$$\mathbf{a}_{s_1 s_2} = (a^1_{s_1 s_2}, \ldots, a^{r_1}_{s_1 s_2}) = \sum_{s_0 \in S_0} G_1(s_0, s_1, s_2) p(s_0) \in \mathbb{R}^{r_1} \tag{8}$$

$$\mathbf{b}_{s_1 s_2} = (b^1_{s_1 s_2}, \ldots, b^{r_2}_{s_1 s_2}) = \sum_{s_0 \in S_0} G_2(s_0, s_1, s_2) p(s_0) \in \mathbb{R}^{r_2}. \tag{9}$$

Now, let $\mathbf{A}$ and $\mathbf{B}$ denote $|S_1| \times |S_2|$ matrices such that elements of the s_1th row and s_2th column are the above defined vectors $\mathbf{a}_{s_1 s_2}$ and $\mathbf{b}_{s_1 s_2}$, respectively. Then, for given realization plans $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ of players 1 and 2, the vectors of expected payoffs of them are represented by

$$\mathbf{x}^T\mathbf{A}\mathbf{y} \triangleq \left(\sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} x_{s_1} a^1_{s_1 s_2} y_{s_2}, \ldots, \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} x_{s_1} a^{r_1}_{s_1 s_2} y_{s_2} \right) \tag{10}$$

$$\mathbf{x}^T\mathbf{B}\mathbf{y} \triangleq \left(\sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} x_{s_1} b^1_{s_1 s_2} y_{s_2}, \ldots, \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} x_{s_1} b^{r_2}_{s_1 s_2} y_{s_2} \right), \tag{11}$$

respectively.

2.2 Nondominated solutions to a multiobjective mathematical programming problem

Before examining nondominated equilibrium solutions in multiobjective two-person nonzero-sum games in extensive form, we first review solutions concepts and related matters in multiobjective mathematical programming. For convenience, let us introduce the following notation: for any two vectors $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^N$, $\mathbf{z} = \mathbf{z}' \Leftrightarrow z_i = z_i{}', i = 1, \ldots, N$; $\mathbf{z} \leqq \mathbf{z}' \Leftrightarrow z_i \leqq z_i{}', i = 1, \ldots, N$; $\mathbf{z} < \mathbf{z}' \Leftrightarrow z_i < z_i{}', i = 1, \ldots, N$; $\mathbf{z} \leq \mathbf{z}' \Leftrightarrow \mathbf{z} \leqq \mathbf{z}'$ and $\mathbf{z} \neq \mathbf{z}'$.

Let $\mathbf{z}$ be an N-dimension real decision variable. Consider a multiobjective mathematical programming problem minimizing K objective functions $\mathbf{f}(\mathbf{z}) = (f_1(\mathbf{z}), \ldots, f_K(\mathbf{z}))^T$ subject to M_1 inequality constraints $\mathbf{g}(\mathbf{z}) = (g_1(\mathbf{z}), \ldots, g_{M_1}(\mathbf{z}))^T \leqq \mathbf{0}$ and M_2 equality constraints $\mathbf{h}(\mathbf{z}) = (h_1(\mathbf{z}), \ldots, h_{M_2}(\mathbf{z}))^T = \mathbf{0}$, where $\mathbf{0}$ is an appropriate dimensional zero vector $(0, \ldots, 0)^T$ corresponding to a dimension of the left hand side. Then, a multiobjective mathematical programming problem can be written as:

$$\text{minimize} \quad \mathbf{f}(\mathbf{z}) \tag{12a}$$

$$\text{subject to} \quad \mathbf{z} \in Z \triangleq \{\mathbf{z} \in \mathbb{R}^N \mid \mathbf{g}(\mathbf{z}) \leqq \mathbf{0}, \quad \mathbf{h}(\mathbf{z}) = \mathbf{0}\}. \tag{12b}$$

Let $O = \{\mathbf{f}(\mathbf{z}) \in \mathbb{R}^K \mid \mathbf{z} \in Z\}$ be a feasible area of the multiple objective values in an objective space.

There does not generally exist a solution minimizing all the objectives simultaneously. Then, Pareto optimal solutions such that any improvement of one objective can be achieved only at the expense of another are introduced, and they are defined as follows.

Definition 1 $\mathbf{z}^* \in Z$ is said to be a Pareto optimal solution if there does not exist another $\mathbf{z} \in Z$ such that

$$\mathbf{f}(\mathbf{z}) \leq \mathbf{f}(\mathbf{z}^*). \tag{13}$$

As a slightly weaker solution concept than Pareto optimality, weak Pareto optimal solutions are also defined by replacing $\leq$ with $<$ in (13).

Next, we provide a definition of a nondominated solution proposed by Yu (1974) which is a solution concept generalized from a Pareto optimal solution. To begin with, we give definitions of a cone and related concepts. A set Λ is said to be a cone if, for any vector $\mathbf{u} \in \Lambda$ and a nonnegative scalar $\eta \geqq 0$, $\eta\mathbf{u} \in \Lambda$ holds. Λ is a convex cone if, for any two vectors $\mathbf{u}_1, \mathbf{u}_2 \in \Lambda$ and two nonnegative scalars $\eta_1, \eta_2 \geqq 0$, $\eta_1\mathbf{u}_1 + \eta_2\mathbf{u}_2 \in \Lambda$ holds. A polar cone of Λ is given as

$$\Lambda^* = \{\gamma \in \mathbb{R}^K \mid \gamma^T\mathbf{u} \leqq 0,\ \forall\mathbf{u} \in \Lambda\}. \tag{14}$$

We define a domination cone prescribing a preference relation. For $\mathbf{o}, \mathbf{o}' \in O \subset \mathbb{R}^K$, when $\mathbf{o}$ is preferred to $\mathbf{o}'$, it is denoted by $\mathbf{o} \succ \mathbf{o}'$. Then, a domination cone is defined as follows.

Definition 2 Given $\mathbf{o} \in O \subset \mathbb{R}^K$, a nonzero vector $\mathbf{d} \in \mathbb{R}^K$ is a domination factor for $\mathbf{o}$ if $\mathbf{o} \succ \mathbf{o} + \rho\mathbf{d}$ for all $\rho > 0$. Then, a domination cone $D(\mathbf{o})$ of $\mathbf{o}$ is a set of all domination factors for $\mathbf{o}$.

Throughout this paper, we use only a constant domination cone $\Lambda \triangleq D(\mathbf{o})$ for all $\mathbf{o} \in O$, and simply call Λ a domination cone. Furthermore, we restrict a domination cone to a polyhedral cone with nonempty interior which can be represented in the following by using its generator $\hat{V} = \{\hat{\mathbf{v}}^t \mid t = 1, \ldots, p\}$:

$$\Lambda = \left\{\pi \in \mathbb{R}^K \;\middle|\; \pi = \sum_{t=1}^{p} \tau_t\hat{\mathbf{v}}^t,\ \tau_t \geqq 0,\ t = 1, \ldots, p\right\}. \tag{15}$$

Then, a multiobjective mathematical programming problem can be defined by the three tuple $(Z, \mathbf{f}(\mathbf{z}), \Lambda)$, where $Z = \{\mathbf{z} \in \mathbb{R}^N \mid \mathbf{g}(\mathbf{z}) \leqq \mathbf{0}, \mathbf{h}(\mathbf{z}) = \mathbf{0}\}$ is a feasible region, $\mathbf{f}(\mathbf{z})$ is a vector of the multiple objectives, and $\Lambda \subset \mathbb{R}^K$ is a domination cone. A nondominated solution to a multiobjective mathematical programming problem $(Z, \mathbf{f}(\mathbf{z}), \Lambda)$ is defined as follows.

Definition 3 Given a multiobjective mathematical programming problem $(Z, \mathbf{f}(\mathbf{z}), \Lambda)$, $\mathbf{z}^* \in Z$ is said to be a nondominated solution if there does not exist another $\mathbf{z} \in Z$ such that

$$\mathbf{f}(\mathbf{z}^*) \in \mathbf{f}(\mathbf{z}) + \Lambda \ \text{ and } \ \mathbf{f}(\mathbf{z}) \neq \mathbf{f}(\mathbf{z}^*). \tag{16}$$

If a domination cone Λ is the negative quadrant, any nondominated solution is also a Pareto optimal solution.

A condition for being a nondominated solution is given by Yu (1974) and Tamura and Miura (1979). Because we restrict a domination cone to a polyhedral cone and the Tamura and Miura condition is a more natural extension of the Kuhn and Tucker condition (Kuhn and Tucker 1951) of optimality for a multiobjective mathematical programming problem, we employ the Tamura and Miura condition to develop a condition for being a nondominated equilibrium solution.

A polar cone Λ^* for a domination cone can be represented in the following by using its generator $V = \{\mathbf{v}^t \mid t = 1, \ldots, q\}$:

$$\Lambda^* = \left\{ \omega \in \mathbb{R}^K \;\middle|\; \omega = \sum_{t=1}^{q} \zeta_t \mathbf{v}^t,\ \zeta_t \geqq 0,\ t = 1, \ldots, q \right\}. \tag{17}$$

Let

$$F(\mathbf{z}) = [\nabla \mathbf{f}(\mathbf{z})^T \mathbf{v}^1, \ldots, \nabla \mathbf{f}(\mathbf{z})^T \mathbf{v}^q], \tag{18}$$

where, for $t \in \{1, \ldots, q\}$,

$$\nabla \mathbf{f}(\mathbf{z})^T \mathbf{v}^t = \begin{bmatrix} \frac{\partial f_1(\mathbf{z})}{\partial z_1} & \cdots & \frac{\partial f_K(\mathbf{z})}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1(\mathbf{z})}{\partial z_N} & \cdots & \frac{\partial f_K(\mathbf{z})}{\partial z_N} \end{bmatrix} \begin{bmatrix} v_1^t \\ \vdots \\ v_K^t \end{bmatrix}. \tag{19}$$

For a multiobjective mathematical programming problem $(Z, \mathbf{f}(\mathbf{z}), \Lambda)$, assume that $\mathbf{g}(\mathbf{z})$ and $\mathbf{h}(\mathbf{z})$ satisfy the Slater constraint qualification, ${\mathbf{v}^t}^T \mathbf{f}(\mathbf{z})$, $t = 1, \ldots, q$ are concave, and Z is a convex set. Then, the following necessary and sufficient condition is given by Tamura and Miura (1979). $\mathbf{z} \in Z$ is a nondominated solution if and only if there exist vectors $\boldsymbol{\mu} \geq \mathbf{0}$, $\boldsymbol{\lambda} \geqq \mathbf{0}$ and $\boldsymbol{\psi}$ such that

$$F(\mathbf{z})\boldsymbol{\mu} - \nabla \mathbf{g}(\mathbf{z})^T \boldsymbol{\lambda} - \nabla \mathbf{h}(\mathbf{z})^T \boldsymbol{\psi} = \mathbf{0} \tag{20a}$$

$$\mathbf{g}(\mathbf{z})^T \boldsymbol{\lambda} = 0 \tag{20b}$$

$$\mathbf{g}(\mathbf{z}) \leqq \mathbf{0} \tag{20c}$$

$$\mathbf{h}(\mathbf{z}) = \mathbf{0}. \tag{20d}$$

If the generator of the polar cone of the domination cone is specified by $V^0 = \{\mathbf{v}^1 = (1, 0, \ldots, 0)^T, \mathbf{v}^2 = (0, 1, 0, \ldots, 0)^T, \ldots, \mathbf{v}^K = (0, \ldots, 0, 1)^T\}$, the Tamura and Miura condition becomes

$$\nabla \mathbf{f}(\mathbf{z})^T \boldsymbol{\mu} - \nabla \mathbf{g}(\mathbf{z})^T \boldsymbol{\lambda} - \nabla \mathbf{h}(\mathbf{z})^T \boldsymbol{\psi} = \mathbf{0} \tag{21a}$$

$$\mathbf{g}(\mathbf{z})^T \boldsymbol{\lambda} = 0 \tag{21b}$$

$$\mathbf{g}(\mathbf{z}) \leqq \mathbf{0} \tag{21c}$$

$$\mathbf{h}(\mathbf{z}) = \mathbf{0}, \tag{21d}$$

and the above condition corresponds to the Kuhn and Tucker condition (Kuhn and Tucker 1951) for Pareto optimality to a multiobjective mathematical programming problem.

2.3 Nondominated equilibrium solutions of a multiobjective two-person nonzero-sum game in extensive form

First, in a multiobjective two-person nonzero-sum game in extensive form, we give a solution concept of Pareto equilibrium solutions, and then extend it to that of nondominated equilibrium solutions by using domination cones.

Definition 4 In a multiobjective two-person nonzero-sum game in extensive form, a pair of realization plans $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is said to be a Pareto equilibrium solution if there does not exist another $(\mathbf{x}, \mathbf{y}) \in X \times Y$ such that

$$\mathbf{x}^{*T}\mathbf{A}\mathbf{y}^* \leq \mathbf{x}^T\mathbf{A}\mathbf{y}^* \tag{22a}$$

$$\mathbf{x}^{*T}\mathbf{B}\mathbf{y}^* \leq \mathbf{x}^{*T}\mathbf{B}\mathbf{y}. \tag{22b}$$

A multiobjective two-person nonzero-sum game in extensive form can be reduced to a single-objective two-person nonzero-sum game by using a weighting coefficient vector $(\mathbf{w}, \mathbf{v}) \in \mathbb{R}^{r_1}_{++} \times \mathbb{R}^{r_2}_{++}$, where $\mathbb{R}^{r_i}_{++} = \{\mathbf{z} \in \mathbb{R}^{r_i} \mid \mathbf{z} > \mathbf{0}\}$, $i = 1, 2$. Furthermore, because the single-objective game in extensive form can be transformed into a game in normal form and there exists at least one equilibrium solution in the game in normal form, in general there exists at least one Pareto equilibrium solution in a multiobjective two-person nonzero-sum game in extensive form (Krieger 2003).

For simplicity, let $\mathbf{f}^1(\mathbf{x}; \mathbf{y}) \triangleq \mathbf{x}^T\mathbf{A}\mathbf{y}$ and $\mathbf{f}^2(\mathbf{y}; \mathbf{x}) \triangleq \mathbf{x}^T\mathbf{B}\mathbf{y}$, and we define nondominated equilibrium solutions in the following.

Definition 5 Let Λ^1 and Λ^2 denote domination cones of players 1 and 2, respectively. Then, in a multiobjective two-person nonzero-sum game in extensive form, a pair of realization plans $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is said to be a nondominated equilibrium solution if there does not exist another $(\mathbf{x}, \mathbf{y}) \in X \times Y$ such that

$$\mathbf{f}^1(\mathbf{x}^*; \mathbf{y}^*) \in \mathbf{f}^1(\mathbf{x}; \mathbf{y}^*) + \Lambda^1, \tag{23a}$$

$$\mathbf{f}^2(\mathbf{y}^*; \mathbf{x}^*) \in \mathbf{f}^2(\mathbf{y}; \mathbf{x}^*) + \Lambda^2. \tag{23b}$$

Especially, by letting $\Lambda^1 = \mathbb{R}^{r_1}_{-}$ and $\Lambda^2 = \mathbb{R}^{r_2}_{-}$ where $\mathbb{R}^{r_i}_{-} = \{\mathbf{z} \in \mathbb{R}^{r_i} \mid \mathbf{z} \leqq \mathbf{0}\}$, $i = 1, 2$, any nondominated equilibrium solution with respect to the domination cones $\mathbb{R}^{r_1}_{-}$ and $\mathbb{R}^{r_2}_{-}$ is also a Pareto equilibrium solution.

The above definition means that $\mathbf{x}^*$ is a nondominated response of player 1 for a strategy $\mathbf{y}^*$ of player 2, and $\mathbf{y}^*$ is a nondominated response of player 2 for a strategy $\mathbf{x}^*$ of player 1. This can be explicitly expressed as follows. The sets of nondominated responses of players 1 and 2 are defined as

$$N^1(\mathbf{y}, \Lambda^1) = \{\mathbf{x} \in X \mid \text{there does not exist } \mathbf{x}' \in X \text{ such that } \mathbf{f}^1(\mathbf{x}; \mathbf{y}) \in \mathbf{f}^1(\mathbf{x}'; \mathbf{y}) + \Lambda^1\}, \tag{24a}$$

$$N^2(\mathbf{x}, \Lambda^2) = \{\mathbf{y} \in Y \mid \text{there does not exist } \mathbf{y}' \in Y \text{ such that } \mathbf{f}^2(\mathbf{y}; \mathbf{x}) \in \mathbf{f}^2(\mathbf{y}'; \mathbf{x}) + \Lambda^2\}. \tag{24b}$$

Then, by using the concept of nondominated responses, the set $N(\Lambda^1, \Lambda^2)$ of nondominated equilibrium solutions can be represented by

$$N(\Lambda^1, \Lambda^2) = \{(\mathbf{x}^*, \mathbf{y}^*) \mid \mathbf{x}^* \in N^1(\mathbf{y}^*, \Lambda^1),\ \mathbf{y}^* \in N^2(\mathbf{x}^*, \Lambda^2)\}. \tag{25}$$

A relation between the domination cones and the sets of nondominated equilibrium solutions is shown in the following proposition.

Proposition 1 *Let Λ^1 and $\Lambda^{1'}$ denote domination cones of player 1, and Λ^2 and $\Lambda^{2'}$ denote domination cones of player 2 in a multiobjective two-person nonzero-sum game in extensive form. Then, if $\Lambda^1 \subset \Lambda^{1'}$ and $\Lambda^2 \subset \Lambda^{2'}$, $N(\Lambda^{1'}, \Lambda^{2'}) \subset N(\Lambda^1, \Lambda^2)$.*

Proof If $\Lambda^1 \subset \Lambda^{1'}$ and $\Lambda^2 \subset \Lambda^{2'}$, from (24a) and (24b), we have

$$N^1(\mathbf{y}, \Lambda^{1'}) \subset N^1(\mathbf{y}, \Lambda^1),$$

$$N^2(\mathbf{x}, \Lambda^{2'}) \subset N^2(\mathbf{x}, \Lambda^2),$$

and from (25), we have

$$N(\Lambda^{1'}, \Lambda^{2'}) \subset N(\Lambda^1, \Lambda^2).$$

□

From the fact that there exists at least one Pareto equilibrium solution (Krieger 2003), we obtain the following theorem showing the existence of nondominated equilibrium solutions.

Theorem 1 *In a multiobjective two-person nonzero-sum game in extensive form, for any domination cones of players 1 and 2, there exists at least one nondominated equilibrium solution.*

Proof Let Λ^1 and Λ^2 denote domination cones of players 1 and 2, respectively. From Definition 3, a pair of realization plans $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is a nondominated equilibrium solution if there does not exist another $(\mathbf{x}, \mathbf{y}) \in X \times Y$ such that

$$\mathbf{x}^{*T}\mathbf{A}\mathbf{y}^* - \mathbf{x}^T\mathbf{A}\mathbf{y}^* \in \Lambda^1,$$
$$\mathbf{x}^{*T}\mathbf{B}\mathbf{y}^* - \mathbf{x}^{*T}\mathbf{B}\mathbf{y} \in \Lambda^2.$$

Let $V^1 = \{\mathbf{v}^{t_1} \mid t_1 = 1, \ldots, q_1\}$ and $W^2 = \{\mathbf{w}^{t_2} \mid t_2 = 1, \ldots, q_2\}$ denote generators of polar cones Λ^{1*} and Λ^{2*} of the domination cones Λ^1 and Λ^2 of players 1 and 2, respectively. Then, we have

$$\mathbf{v}^{t_1 T}(\mathbf{x}^{*T}\mathbf{A}\mathbf{y}^* - \mathbf{x}^T\mathbf{A}\mathbf{y}^*) \leqq 0, \quad t_1 = 1, \ldots, q_1,$$
$$\mathbf{w}^{t_2 T}(\mathbf{x}^{*T}\mathbf{B}\mathbf{y}^* - \mathbf{x}^{*T}\mathbf{B}\mathbf{y}) \leqq 0, \quad t_2 = 1, \ldots, q_2,$$

and they can be rewritten as

$$\mathbf{x}^{*T}(\mathbf{v}^{t_1 T}\mathbf{A})\mathbf{y}^* - \mathbf{x}^T(\mathbf{v}^{t_1 T}\mathbf{A})\mathbf{y}^* \leqq 0, \quad t_1 = 1, \ldots, q_1,$$
$$\mathbf{x}^{*T}(\mathbf{w}^{t_2 T}\mathbf{B})\mathbf{y}^* - \mathbf{x}^{*T}(\mathbf{w}^{t_2 T}\mathbf{B})\mathbf{y} \leqq 0, \quad t_2 = 1, \ldots, q_2.$$

Let $\mathbf{A}' \triangleq (A'_1, \ldots, A'_{q_1})^T$, $A'_{t_1} = \mathbf{v}^{t_1 T}\mathbf{A}$, $t_1 = 1, \ldots, q_1$ and $\mathbf{B}' \triangleq (B'_1, \ldots, B'_{q_2})^T$, $B'_{t_2} = \mathbf{w}^{t_2 T}\mathbf{B}$, $t_2 = 1, \ldots, q_2$. Then, if there does not exist another $(\mathbf{x}, \mathbf{y}) \in X \times Y$ such that

$$\mathbf{x}^{*T}\mathbf{A}'\mathbf{y}^* \leq \mathbf{x}^T\mathbf{A}'\mathbf{y}^*,$$
$$\mathbf{x}^{*T}\mathbf{B}'\mathbf{y}^* \leq \mathbf{x}^{*T}\mathbf{B}'\mathbf{y},$$

a pair of realization plans $(\mathbf{x}^*, \mathbf{y}^*)$ is a nondominated equilibrium solution and at the same time it is a Pareto equilibrium solution of the multiobjective two-person nonzero-sum game in extensive form. Therefore, because there exists at least one Pareto equilibrium solution, there also exists at least one nondominated equilibrium solution. □

2.4 Necessary and sufficient condition for a nondominated equilibrium solution

In a multiobjective two-person nonzero-sum game in extensive form, given domination cones Λ^1 and Λ^2 of players 1 and 2, respectively, the fact that a realization plan $\mathbf{x}^*$ of player 1 is a nondominated response for a realization plan $\mathbf{y}^*$ of player 2 corresponds to the fact that $\mathbf{x}^*$ is a nondominated solution to a multiobjective mathematical programming problem $(X, \mathbf{f}^1(\mathbf{x}; \mathbf{y}^*), \Lambda^1)$, and similarly the fact that a realization plan $\mathbf{y}^*$ of player 2 is a nondominated response for a realization plan $\mathbf{x}^*$ of player 1 corresponds to the fact that $\mathbf{y}^*$ is a nondominated solution to a multiobjective mathematical programming problem $(Y, \mathbf{f}^2(\mathbf{y}; \mathbf{x}^*), \Lambda^2)$.

Assume that X, Y, $\mathbf{f}^1(\mathbf{x};\mathbf{y}^*) = \mathbf{x}^T\mathbf{A}\mathbf{y}^*$, and $\mathbf{f}^2(\mathbf{y};\mathbf{x}^*) = \mathbf{x}^{*T}\mathbf{B}\mathbf{y}$ are represented by (6), (7), (10), and (11), respectively, and Λ^1 and Λ^2 are polyhedral domination cones such as (15). Then, the following theorem can be obtained by using the Tamura and Miura condition (20) to the two multiobjective mathematical programming problems $(X, \mathbf{f}^1(\mathbf{x};\mathbf{y}^*), \Lambda^1)$ and $(Y, \mathbf{f}^2(\mathbf{y};\mathbf{x}^*), \Lambda^2)$.

Theorem 2 *In a multiobjective two-person nonzero-sum game in extensive form, let* $V^1 = \{\mathbf{v}^{t_1} \mid t_1 = 1,\ldots,q_1\}$ *and* $W^2 = \{\mathbf{w}^{t_2} \mid t_2 = 1,\ldots,q_2\}$ *denote generators of polar cones* Λ^{1*} *and* Λ^{2*} *of the domination cones* Λ^1 *and* Λ^2 *of players 1 and 2, respectively, where* Λ^{1*} *and* Λ^{2*} *are represented as*

$$\Lambda^{1*} = \left\{ \boldsymbol{\omega}^1 \in \mathbb{R}^{r_1} \;\middle|\; \boldsymbol{\omega}^1 = \sum_{t_1=1}^{q_1} \delta_{t_1}\mathbf{v}^{t_1},\, \delta_{t_1} \geqq 0,\, t_1 = 1,\ldots,q_1 \right\}, \tag{26a}$$

$$\Lambda^{2*} = \left\{ \boldsymbol{\omega}^2 \in \mathbb{R}^{r_2} \;\middle|\; \boldsymbol{\omega}^2 = \sum_{t_2=1}^{q_2} \varepsilon_{t_2}\mathbf{w}^{t_2},\, \varepsilon_{t_2} \geqq 0,\, t_2 = 1,\ldots,q_2 \right\}. \tag{26b}$$

Then, $(\mathbf{x}^*, \mathbf{y}^*)$ is a nondominated equilibrium solution if and only if there exist α^*, β^*, δ^*, and ε^* satisfying the following condition, which are $|U_1|$-, $|U_2|$-, q_1-, and q_2-dimensional vectors, respectively.

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|} \delta^*_{t_1} v^{t_1}_{k_1} x^*_{s_1} a^{k_1}_{s_1 s_2} y^*_{s_2} - \sum_{s_1=1}^{|S_1|}\sum_{u_1=0}^{|U_1|} \alpha^*_{u_1} e^1_{u_1 s_1} x^*_{s_1} = 0 \tag{27a}$$

$$\sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|} \varepsilon^*_{t_2} w^{t_2}_{k_2} x^*_{s_1} b^{k_2}_{s_1 s_2} y^*_{s_2} - \sum_{s_2=1}^{|S_2|}\sum_{u_2=0}^{|U_2|} \beta^*_{u_2} e^2_{u_2 s_2} y^*_{s_2} = 0 \tag{27b}$$

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|} \delta^*_{t_1} v^{t_1}_{k_1} a^{k_1}_{s_1 s_2} y^*_{s_2} - \sum_{u_1=0}^{|U_1|} \alpha^*_{u_1} e^1_{u_1 s_1} \leqq 0, \quad s_1 = 1,\ldots,|S_1| \tag{27c}$$

$$\sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|} \varepsilon^*_{t_2} w^{t_2}_{k_2} x^*_{s_1} b^{k_2}_{s_1 s_2} - \sum_{u_2=0}^{|U_2|} \beta^*_{u_2} e^2_{u_2 s_2} \leqq 0, \quad s_2 = 1,\ldots,|S_2| \tag{27d}$$

$$\sum_{s_1=1}^{|S_1|} e^1_{u_1 s_1} x^*_{s_1} - e^1_{u_1} = 0, \quad u_1 = 0,\ldots,|U_1| \tag{27e}$$

$$\sum_{s_2=1}^{|S_2|} e^2_{u_2 s_2} y^*_{s_2} - e^2_{u_2} = 0, \quad u_2 = 0,\ldots,|U_2| \tag{27f}$$

$$\mathbf{x}^* \geqq \mathbf{0} \tag{27g}$$

$$\mathbf{y}^* \geqq \mathbf{0} \tag{27h}$$

$$\boldsymbol{\delta}^* \geq \mathbf{0} \tag{27i}$$

$$\boldsymbol{\varepsilon}^* \geq \mathbf{0} \tag{27j}$$

Proof Let the expected payoffs and the inequality and equality constraints of player 1 be expressed as

$$\mathbf{f}^1(\mathbf{x};\mathbf{y}^*) = \mathbf{x}^T\mathbf{A}\mathbf{y}^* = \left(\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|} x_{s_1} a^1_{s_1 s_2} y^*_{s_2}, \ldots, \sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|} x_{s_1} a^{r_1}_{s_1 s_2} y^*_{s_2}\right)$$

$$h^1_{u_1}(\mathbf{x}) = \sum_{s_1=1}^{|S_1|} e^1_{u_1 s_1} x_{s_1} - e^1_{u_1} = 0, \quad u_1 = 0, \ldots, |U_1|$$

$$g^1_{s_1}(\mathbf{x}) = -x_{s_1} \leqq 0, \quad s_1 = 1, \ldots, |S_1|,$$

and $X = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{g}^1(\mathbf{x}) \leqq \mathbf{0},\ h^1(\mathbf{x}) = 0\}$. Similarly, let the expected payoffs and the inequality and equality constraints of player 2 be expressed as

$$\mathbf{f}^2(\mathbf{y}) = \mathbf{x}^{*T}\mathbf{B}\mathbf{y} = \left(\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|} x^*_{s_1} b^1_{s_1 s_2} y_{s_2}, \ldots, \sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|} x^*_{s_1} b^{r_2}_{s_1 s_2} y_{s_2}\right)$$

$$h^2_{u_2}(\mathbf{y}) = \sum_{s_2=1}^{|S_2|} e^2_{u_2 s_2} y_{s_2} - e^2_{u_2} = 0, \quad u_2 = 0, \ldots, |U_2|$$

$$g^2_{s_2}(\mathbf{y}) = -y_{s_2} \leqq 0, \quad s_2 = 1, \ldots, |S_2|,$$

and $Y = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{g}^2(\mathbf{y}) \leqq \mathbf{0},\ h^2(\mathbf{y}) = 0\}$. Because the generators of polar cones Λ^{1*} and Λ^{2*} of the domination cones Λ^1 and Λ^2 of players 1 and 2 are $V^1 = \{\mathbf{v}^{t_1} \mid t_1 = 1, \ldots, q_1\}$ and $V^2 = \{\mathbf{w}^{t_2} \mid t_2 = 1, \ldots, q_2\}$, respectively, let

$$F^1(\mathbf{x}) = \left[\nabla\mathbf{f}^1(\mathbf{x};\mathbf{y}^*)^T\mathbf{v}^1, \ldots, \nabla\mathbf{f}^1(\mathbf{x};\mathbf{y}^*)^T\mathbf{v}^{q_1}\right],$$
$$F^2(\mathbf{y}) = \left[\nabla\mathbf{f}^2(\mathbf{y};\mathbf{x}^*)^T\mathbf{w}^1, \ldots, \nabla\mathbf{f}^2(\mathbf{y};\mathbf{x}^*)^T\mathbf{w}^{q_2}\right].$$

Because $\mathbf{v}^{t_1\,T}\mathbf{f}^1(\mathbf{x};\mathbf{y}^*)$, $t_1 = 1, \ldots, q_1$, and $\mathbf{w}^{t_2\,T}\mathbf{f}^2(\mathbf{y};\mathbf{x}^*)$, $t_2 = 1, \ldots, q_2$ are concave, and X and Y are convex sets, the two multiobjective mathematical programming problems $(X, \mathbf{f}^1(\mathbf{x};\mathbf{y}^*), \Lambda^1)$ and $(Y, \mathbf{f}^2(\mathbf{y};\mathbf{x}^*), \Lambda^2)$ satisfy the assumption of the theorem by Tamura and Miura (1979). Let $\boldsymbol{\delta}^*$, $\boldsymbol{\lambda}^*$, and $\boldsymbol{\alpha}^*$ be multiplier vectors for $F(\mathbf{z})$, $\nabla\mathbf{g}(\mathbf{z})$ and $\nabla\mathbf{h}(\mathbf{z})$ in (20), respectively, and then from (20a), we have

$$F^1(\mathbf{x}^*)\boldsymbol{\delta}^* - \nabla\mathbf{g}^1(\mathbf{x}^*)^T\boldsymbol{\lambda}^* - \nabla h^1(\mathbf{x}^*)\boldsymbol{\alpha}^* = \mathbf{0}.$$

It can be rewritten as

$$\lambda^*_{s_1} = -\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|} \delta^*_{t_1} v^{t_1}_{k_1} a^{k_1}_{s_1 s_2} y^*_{s_2} + \sum_{u_1=0}^{|U_1|} \alpha^*_{u_1} e^1_{u_1 s_1}, \quad s_1 = 1, \ldots, |S_1| \tag{28}$$

From (20b), it follows that

$$\mathbf{g}^1(\mathbf{x}^*)^T\boldsymbol{\lambda}^* = 0.$$

Substitution in (28) yields (27a). Because $\boldsymbol{\lambda}^* \geqq \mathbf{0}$, from (28), one finds (27c). From the Tamura and Miura condition, we directly obtain (27e), (27g), and (27i).

For player 2, by a similar procedure, we have (27b), (27d), (27f), (27h), and (27j). □

If the domination cones of players 1 and 2 are the negative quadrant, any nondominated equilibrium solution with respect to the domination cones is also a weak Pareto equilibrium solution and the generators of the polar cones of the domination cone are $V^1 = \{\mathbf{v}^1 = (1, 0, \ldots, 0)^T, \mathbf{v}^2 = (0, 1, 0, \ldots, 0)^T, \ldots, \mathbf{v}^{r_1} = (0, \ldots, 0, 1)^T\}$ and $W^2 = \{\mathbf{w}^1 = (1, 0, \ldots, 0)^T, \mathbf{w}^2 = (0, 1, 0, \ldots, 0)^T, \ldots, \mathbf{w}^{r_2} = (0, \ldots, 0, 1)^T\}$. Furthermore, if the multiplier vectors are strictly positive, i.e., $\boldsymbol{\delta} > \mathbf{0}$, $\boldsymbol{\varepsilon} > \mathbf{0}$, any nondominated equilibrium solution is also a Pareto equilibrium solution. From the above facts, we straightforwardly obtain the following corollary.

Corollary 1 *For a multiobjective two-person nonzero-sum game in extensive form,* $(\mathbf{x}^*, \mathbf{y}^*)$ *is a Pareto equilibrium solution if and only if there exist* $\boldsymbol{\alpha}^*$, $\boldsymbol{\beta}^*$, $\boldsymbol{\delta}^*$, *and* $\boldsymbol{\varepsilon}^*$ *satisfying the following condition.*

$$\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta^*_{k_1}x^*_{s_1}a^{k_1}_{s_1s_2}y^*_{s_2} - \sum_{s_1=1}^{|S_1|}\sum_{u_1=0}^{|U_1|}\alpha^*_{u_1}e^1_{u_1s_1}x^*_{s_1} = 0 \tag{29a}$$

$$\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon^*_{k_2}x^*_{s_1}b^{k_2}_{s_1s_2}y^*_{s_2} - \sum_{s_2=1}^{|S_2|}\sum_{u_2=0}^{|U_2|}\beta^*_{u_2}e^2_{u_2s_2}y^*_{s_2} = 0 \tag{29b}$$

$$\sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|}\delta^*_{k_1}a^{k_1}_{s_1s_2}y^*_{s_2} - \sum_{u_1=0}^{|U_1|}\alpha^*_{u_1}e^1_{u_1s_1} \leqq 0, \quad s_1 = 1, \ldots, |S_1| \tag{29c}$$

$$\sum_{k_2=1}^{r_2}\sum_{s_2=1}^{|S_2|}\varepsilon^*_{k_2}x^*_{s_1}b^{k_2}_{s_1s_2} - \sum_{u_2=0}^{|U_2|}\beta^*_{u_2}e^2_{u_2s_2} \leqq 0, \quad s_2 = 1, \ldots, |S_2| \tag{29d}$$

$$\sum_{s_1=1}^{|S_1|}e^1_{u_1s_1}x^*_{s_1} - e^1_{u_1} = 0, \quad u_1 = 0, \ldots, |U_1| \tag{29e}$$

$$\sum_{s_2=1}^{|S_2|}e^2_{u_2s_2}y^*_{s_2} - e^2_{u_2} = 0, \quad u_2 = 0, \ldots, |U_2| \tag{29f}$$

$$\mathbf{x}^* \geqq \mathbf{0} \tag{29g}$$

$$\mathbf{y}^* \geqq \mathbf{0} \tag{29h}$$

$$\boldsymbol{\delta}^* > \mathbf{0} \tag{29i}$$

$$\boldsymbol{\varepsilon}^* > \mathbf{0} \tag{29j}$$

2.5 Nondominated equilibrium solutions and corresponding mathematical programming problem

Using the necessary and sufficient condition for a nondominated equilibrium solution, we formulate a mathematical programming problem whose optimal solutions are nondominated equilibrium solutions.

Theorem 3 *In a multiobjective two-person nonzero-sum game in extensive form, let* $V^1 = \{\mathbf{v}^{t_1} \mid t_1 = 1, \ldots, q_1\}$ *and* $W^2 = \{\mathbf{w}^{t_2} \mid t_2 = 1, \ldots, q_2\}$ *denote generators of polar cones* Λ^{1*} *and* Λ^{2*} *of the domination cones* Λ^1 *and* Λ^2 *of players 1 and 2, respectively, where* Λ^{1*} *and* Λ^{2*} *are represented as (26). Then,* $(\mathbf{x}^*, \mathbf{y}^*)$ *is a nondominated equilibrium solution if and only if* $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*, \boldsymbol{\varepsilon}^*)$ *is an optimal solution to the following mathematical programming problem.*

$$\text{maximize} \quad \sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{t_1} v_{k_1}^{t_1} x_{s_1} a_{s_1 s_2}^{k_1} y_{s_2} + \sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{t_2} w_{k_2}^{t_2} x_{s_1} b_{s_1 s_2}^{k_2} y_{s_2}$$

$$- \sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1} e_{u_1 s_1}^{1} x_{s_1} - \sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2} e_{u_2 s_2}^{2} y_{s_2} \tag{30a}$$

$$\text{subject to} \quad \sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|}\delta_{t_1} v_{k_1}^{t_1} a_{s_1 s_2}^{k_1} y_{s_2} - \sum_{u_1=0}^{|U_1|}\alpha_{u_1} e_{u_1 s_1}^{1} \leqq 0, \quad s_1 = 1, \ldots, |S_1| \tag{30b}$$

$$\sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\varepsilon_{t_2} w_{k_2}^{t_2} x_{s_1} b_{s_1 s_2}^{k_2} - \sum_{u_2=0}^{|U_2|}\beta_{u_2} e_{u_2 s_2}^{2} \leqq 0, \quad s_2 = 1, \ldots, |S_2| \tag{30c}$$

$$\sum_{s_1=1}^{|S_1|} e_{u_1 s_1}^{1} x_{s_1} - e_{u_1}^{1} = 0, \quad u_1 = 0, \ldots, |U_1| \tag{30d}$$

$$\sum_{s_2=1}^{|S_2|} e_{u_2 s_2}^{2} y_{s_2} - e_{u_2}^{2} = 0, \quad u_2 = 0, \ldots, |U_2| \tag{30e}$$

$$\mathbf{x} \geqq \mathbf{0} \tag{30f}$$

$$\mathbf{y} \geqq \mathbf{0} \tag{30g}$$

$$\boldsymbol{\delta} \geq \mathbf{0} \tag{30h}$$

$$\boldsymbol{\varepsilon} \geq \mathbf{0}. \tag{30i}$$

Proof By multiplying the constraint (30b) by x_i satisfying the constraints (30d) and (30f) and multiplying the constraint (30c) by y_j satisfying the constraints (30e) and (30g), and then summing up them, we have

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{t_1} v_{k_1}^{t_1} x_{s_1} a_{s_1 s_2}^{k_1} y_{s_2} + \sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{t_2} w_{k_2}^{t_2} x_{s_1} b_{s_1 s_2}^{k_2} y_{s_2}$$

$$- \sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1} e_{u_1 s_1}^{1} x_{s_1} - \sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2} e_{u_2 s_2}^{2} y_{s_2} \leqq 0 \tag{31}$$

From (31), a value of the objective function (30a) is not larger than zero even if optimal.

Because there exists a nondominated equilibrium solution from Theorem 1, let $(\mathbf{x}^*, \mathbf{y}^*)$ be a nondominated equilibrium solution. From Theorem 2, $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*, \boldsymbol{\varepsilon}^*)$ satisfies (27a)–(27j), and therefore it also satisfies the constraints (30b)–(30i) of the problem (30). Moreover, from (27a) and (27b), we have

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{t_1}^* v_{k_1}^{t_1} x_{s_1}^* a_{s_1 s_2}^{k_1} y_{s_2}^* + \sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{t_2}^* w_{k_2}^{t_2} x_{s_1}^* b_{s_1 s_2}^{k_2} y_{s_2}^*$$

$$- \sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1}^* e_{u_1 s_1}^{1} x_{s_1}^* - \sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2}^* e_{u_2 s_2}^{2} y_{s_2}^* = 0$$

and then from (31), $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*, \boldsymbol{\varepsilon}^*)$ is an optimal solution to the problem (30).

Let $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*, \boldsymbol{\varepsilon}^*)$ be an optimal solution to the problem (30). Because, from Theorem 1, there exists a nondominated equilibrium solution and it satisfies the condition

(27a)–(27j) of Theorem 2, there exists at least one feasible solution to the problem (30) holding the equation

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{t_1} v_{k_1}^{t_1} x_{s_1} a_{s_1 s_2}^{k_1} y_{s_2} + \sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{t_2} w_{k_2}^{t_2} x_{s_1} b_{s_1 s_2}^{k_2} y_{s_2} - \sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1} e_{u_1 s_1}^{1} x_{s_1} - \sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2} e_{u_2 s_2}^{2} y_{s_2} = 0.$$

If $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*, \boldsymbol{\varepsilon}^*)$ is an optimal solution to the problem (30), it must also hold

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{t_1}^* v_{k_1}^{t_1} x_{s_1}^* a_{s_1 s_2}^{k_1} y_{s_2}^* + \sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{t_2}^* w_{k_2}^{t_2} x_{s_1}^* b_{s_1 s_2}^{k_2} y_{s_2}^* - \sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1}^* e_{u_1 s_1}^{1} x_{s_1}^* - \sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2}^* e_{u_2 s_2}^{2} y_{s_2}^* = 0. \tag{32}$$

From (30b), (30c), and (32), we have

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{t_1}^* v_{k_1}^{t_1} x_{s_1}^* a_{s_1 s_2}^{k_1} y_{s_2}^* - \sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1}^* e_{u_1 s_1}^{1} x_{s_1}^* = 0$$

$$\sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{t_2}^* w_{k_2}^{t_2} x_{s_1}^* b_{s_1 s_2}^{k_2} y_{s_2}^* - \sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2}^* e_{u_2 s_2}^{2} y_{s_2}^* = 0,$$

and $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*, \boldsymbol{\varepsilon}^*)$ satisfies the condition (27a)–(27j) of Theorem 2. Therefore, $(\mathbf{x}^*, \mathbf{y}^*)$ is a nondominated equilibrium solution. □

By specifying the generators of the polar cones of the domination cone as $V^1 = \{\mathbf{v}^1 = (1, 0, \dots, 0)^T, \mathbf{v}^2 = (0, 1, 0, \dots, 0)^T, \dots, \mathbf{v}^{r_1} = (0, \dots, 0, 1)^T\}$ and $W^2 = \{\mathbf{w}^1 = (1, 0, \dots, 0)^T, \mathbf{w}^2 = (0, 1, 0, \dots, 0)^T, \dots, \mathbf{w}^{r_2} = (0, \dots, 0, 1)^T\}$, for the strictly positive multiplier vectors $\boldsymbol{\delta} > \mathbf{0}$ and $\boldsymbol{\varepsilon} > \mathbf{0}$, any nondominated equilibrium solution with respect to the domination cones is also a Pareto equilibrium solution, and then we obtain the following corollary.

Corollary 2 *For a multiobjective two-person nonzero-sum game in extensive form,* $(\mathbf{x}^*, \mathbf{y}^*)$ *is a Pareto equilibrium solution if and only if* $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\delta}^*, \boldsymbol{\varepsilon}^*)$ *is an optimal solution to the following mathematical programming problem.*

$$\text{maximize} \quad \sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{k_1} x_{s_1} a_{s_1 s_2}^{k_1} y_{s_2} + \sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{k_2} x_{s_1} b_{s_1 s_2}^{k_2} y_{s_2} - \sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1} e_{u_1 s_1}^{1} x_{s_1} - \sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2} e_{u_2 s_2}^{2} y_{s_2} \tag{33a}$$

$$\text{subject to} \quad \sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|}\delta_{k_1} a_{s_1 s_2}^{k_1} y_{s_2} - \sum_{u_1=0}^{|U_1|}\alpha_{u_1} e_{u_1 s_1}^{1} \le 0, \quad s_1 = 1, \dots, |S_1| \tag{33b}$$

$$\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|} \varepsilon_{k_2} x_{s_1} b^{k_2}_{s_1 s_2} - \sum_{u_2=0}^{|U_2|} \beta_{u_2} e^2_{u_2 s_2} \leq 0, \quad s_2 = 1, \ldots, |S_2| \tag{33c}$$

$$\sum_{s_1=1}^{|S_1|} e^1_{u_1 s_1} x_{s_1} - e^1_{u_1} = 0, \quad u_1 = 0, \ldots, |U_1| \tag{33d}$$

$$\sum_{s_2=1}^{|S_2|} e^2_{u_2 s_2} y_{s_2} - e^2_{s_2} = 0, \quad u_2 = 0, \ldots, |U_2| \tag{33e}$$

$$\mathbf{x} \geqq \mathbf{0} \tag{33f}$$

$$\mathbf{y} \geqq \mathbf{0} \tag{33g}$$

$$\boldsymbol{\delta} > \mathbf{0} \tag{33h}$$

$$\boldsymbol{\varepsilon} > \mathbf{0}. \tag{33i}$$

3. A numerical example

We consider a multiobjective two-person nonzero-sum game in extensive form shown in Fig. 1. Both players have two objectives. In this game, a probability distribution of the chance move is given by $\mathbf{p} = (p_1, p_2) = (0.6, 0.4)$. First, player 1 who knows a choice of chance player selects her choice, and then player 2 who does not know the choice of chance player but knows the choice of player 1 determines his choice. Next, player 1 who remembers her action at the previous node but does not know the choice of player 2 selects her choice again.

There are 13 sequences of player 1 and 5 sequences of player 2, and realization plans of players 1 and 2 are represented by

$$\begin{aligned}\mathbf{x} = (&x(\emptyset), x(r_1), x(l_1), x(r_2), x(l_2), x(r_1 r_3), x(r_1 l_3), x(l_1 r_4), x(l_1 l_4), x(r_2 r_5),\\ &x(r_2 l_5), x(l_2 r_6), x(l_2 l_6))^T,\\ \mathbf{y} = (&y(\emptyset), y(c_1), y(d_1), y(c_2), y(d_2))^T,\end{aligned}$$

respectively.

The constraint matrix E^1 of player 1 is given by

$$E^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0&0&0&0&0&0&0&0 \\ -1 & 1 & 1 & 0 & 0 & 0&0&0&0&0&0&0&0 \\ -1 & 0 & 0 & 1 & 1 & 0&0&0&0&0&0&0&0 \\ 0 & -1 & 0 & 0 & 0 & 1&1&0&0&0&0&0&0 \\ 0 & 0 & -1 & 0 & 0 & 0&0&1&1&0&0&0&0 \\ 0 & 0 & 0 & -1 & 0 & 0&0&0&0&1&1&0&0 \\ 0 & 0 & 0 & 0 & -1 & 0&0&0&0&0&0&1&1 \end{bmatrix},$$

and the constraint matrix E^2 of player 2 is also given by

$$E^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Because the probabilities of chance move are $p_1 = 0.6$ and $p_2 = 0.4$, the vector valued payoff matrix **A** of player 1 can be computed as follows.

$$\mathbf{A} = \begin{bmatrix} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & \left(\frac{3}{5},\frac{3}{5}\right) & \left(-\frac{3}{5},-\frac{6}{5}\right) & (0,0) & (0,0) \\ (0,0) & (0,0) & \left(\frac{3}{5},\frac{3}{5}\right) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & \left(\frac{6}{5},\frac{3}{5}\right) & \left(\frac{6}{5},\frac{6}{5}\right) \\ (0,0) & (0,0) & (0,0) & \left(-\frac{3}{5},-\frac{6}{5}\right) & \left(\frac{6}{5},\frac{9}{5}\right) \\ (0,0) & (0,0) & \left(\frac{4}{5},\frac{6}{5}\right) & (0,0) & (0,0) \\ (0,0) & \left(-\frac{2}{5},-\frac{4}{5}\right) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & \left(\frac{2}{5},\frac{4}{5}\right) \\ (0,0) & (0,0) & (0,0) & \left(-\frac{4}{5},-\frac{4}{5}\right) & \left(\frac{2}{5},\frac{2}{5}\right) \end{bmatrix}. \tag{34}$$

The vector valued payoff matrix **B** of player 2 can be also computed as follows.

$$\mathbf{B} = \begin{bmatrix} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & \left(\frac{3}{5},-\frac{3}{5}\right) & \left(-\frac{3}{5},\frac{3}{5}\right) & (0,0) & (0,0) \\ (0,0) & \left(\frac{3}{5},-\frac{3}{5}\right) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & \left(-\frac{6}{5},-\frac{12}{5}\right) \\ (0,0) & (0,0) & (0,0) & (0,0) & \left(\frac{6}{5},\frac{12}{5}\right) \\ (0,0) & \left(\frac{4}{5},\frac{4}{5}\right) & \left(-\frac{4}{5},-\frac{4}{5}\right) & (0,0) & (0,0) \\ (0,0) & \left(\frac{2}{5},\frac{2}{5}\right) & \left(\frac{2}{5},-\frac{6}{5}\right) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & \left(\frac{2}{5},\frac{4}{5}\right) & (0,0) \\ (0,0) & (0,0) & (0,0) & \left(-\frac{2}{5},\frac{2}{5}\right) & (0,0) \end{bmatrix}. \tag{35}$$

Let $\hat{V}^{1P} = \{\hat{\mathbf{v}}^{1P} = (0,-1)^T, \hat{\mathbf{v}}^{2P} = (-1,0)^T\}$ and $\hat{W}^{1P} = \{\hat{\mathbf{w}}^{1P} = (0,-1)^T, \hat{\mathbf{w}}^{2P} = (-1,0)^T\}$ be generators of the domination cones Λ^{1P} and Λ^{2P} of players 1 and 2, respectively, and the corresponding polar cones are represented by

$$\Lambda^{1P} = \left\{\boldsymbol{\omega}^1 \in \mathbb{R}^2 \mid \boldsymbol{\omega}^1 = \delta_1\mathbf{v}^{1P} + \delta_2\mathbf{v}^{2P},\ \boldsymbol{\delta} = (\delta_1,\delta_2) \geq \mathbf{0}\right\},$$
$$\Lambda^{2P} = \left\{\boldsymbol{\omega}^2 \in \mathbb{R}^2 \mid \boldsymbol{\omega}^2 = \varepsilon_1\mathbf{w}^{1P} + \varepsilon_2\mathbf{w}^{2P},\ \boldsymbol{\varepsilon} = (\varepsilon_1,\varepsilon_2) \geq \mathbf{0}\right\},$$

where $\mathbf{v}^{1P} = (1,0)^T$, $\mathbf{v}^{2P} = (0,1)^T$, $\mathbf{w}^{1P} = (1,0)^T$, and $\mathbf{w}^{2P} = (0,1)^T$. Then, any non-dominated equilibrium solution of the game is also a Pareto equilibrium solution, and from Corollary 2, we can formulate the mathematical programming problem yielding Pareto equilibrium solutions. The problem, which is a quadratic programming problem, has 32 variables and 50 constraints.

We develop a software based on the successive quadratic programming method for solving problems (30) and (33). By using the software, we repeatedly solve the formulated problem and obtain the following set $X^P = X_1^P \cup X_2^P \cup X_3^P$ of Pareto equilibrium solutions:

$$X_1^P = \left\{ \left(\left(1, 0, 1, 0, 1, 0, 0, \frac{1}{3}, \frac{2}{3}, 0, 0, 1, 0\right)^T, \left(1, 1-a, a, \frac{1}{4}, \frac{3}{4}\right)^T \right) \middle| \, 0 \leqq a \leqq \frac{1}{2} \right\},$$
$$X_2^P = \left\{ \left((1, 0, 1, b, 1-b, 0, 0, 1, 0, b, 0, 1-b, 0)^T, (1, 1, 0, 1, 0)^T \right) \mid 0 \leqq b \leqq 1 \right\},$$
$$X_3^P = \left\{ \left((1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, c, 1-c)^T, (1, 1-d, d, 1, 0)^T \right) \middle| \, 0 \leqq c \leqq 1, \; 0 \leqq d \leqq \frac{2}{3} \right\}.$$

To obtain the result, we take not more than one minute in a PC with Pentium 4, 3 GHz. In Fig. 2, we depict feasible areas of expected payoffs and areas painted black which corresponds to the Pareto equilibrium solutions in a payoff space of players 1 and 2.

Next, we slightly change domination cones compared with the previous example. Let $\hat{V}^{1N} = \{\hat{\mathbf{v}}^{1N} = (-1, 1)^T, \hat{\mathbf{v}}^{2N} = (1, -2)^T\}$ and $\hat{W}^{1N} = \{\hat{\mathbf{w}}^{1N} = (-1, 1)^T, \hat{\mathbf{w}}^{2N} = (1, -2)^T\}$ be generators of the domination cones Λ^{1N} and Λ^{2N} of players 1 and 2, respectively, and the corresponding polar cones are represented by

$$\Lambda^{1N} = \left\{ \boldsymbol{\omega}^1 \in \mathbb{R}^2 \mid \boldsymbol{\omega}^1 = \delta_1 \mathbf{v}^{1N} + \delta_2 \mathbf{v}^{2N}, \; \delta_1, \delta_2 \geqq 0 \right\},$$
$$\Lambda^{2N} = \left\{ \boldsymbol{\omega}^2 \in \mathbb{R}^2 \mid \boldsymbol{\omega}^2 = \varepsilon_1 \mathbf{w}^{1N} + \varepsilon_2 \mathbf{w}^{2N}, \; \varepsilon_1, \varepsilon_2 \geqq 0 \right\},$$

where $\mathbf{v}^{1N} = (1, 1)^T$, $\mathbf{v}^{2N} = (2, 1)^T$, $\mathbf{w}^{1N} = (1, 1)^T$, and $\mathbf{w}^{2N} = (2, 1)^T$.

From Theorem 3, we formulate a similar mathematical programming problem yielding nondominated equilibrium solutions, and obtain the following set $X^N = X_1^N \cup X_2^N \cup X_3^N$ of nondominated equilibrium solutions:

$$X_1^N = \left\{ \left(\left(1, 0, 1, 0, 1, 0, 0, \frac{1}{3}, \frac{2}{3}, 0, 0, 1, 0\right)^T, \left(1, 1-a, a, \frac{1}{4}, \frac{3}{4}\right)^T \right) \middle| \, 0 \leqq a \leqq \frac{9}{20} \right\},$$
$$X_2^N = X_2^P,$$
$$X_3^N = \left\{ \left((1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0)^T, (1, 1-b, b, 1, 0)^T \right) \middle| \, 0 \leqq b \leqq \frac{3}{5} \right\}.$$

Compared with the set of Pareto equilibrium solutions, we can understand that, from $X_1^N \subset X_1^P$ and $X_3^N \subset X_3^P$, $X^N \subset X^P$.

In Fig. 3, we also depict feasible areas of expected payoffs and areas corresponding to the nondominated equilibrium solutions in the payoff space of players 1 and 2.

Comparing Figs. 2 and 3, we can confirm the inclusion relation between the set of Pareto equilibrium solutions and that of nondominated equilibrium solutions in the payoff space.

4. Conclusions

In this paper, in a multiobjective two-person nonzero-sum game in extensive form, we have defined nondominated equilibrium solutions based on domination cones and have given a necessary and sufficient condition for a pair of realization plans to be a nondominated equilibrium solution by applying the Tamura and Miura condition for multiobjective mathematical programming problems. Using the necessary and sufficient condition, we have formulated a mathematical programming problem whose optimal solutions correspond to nondominated

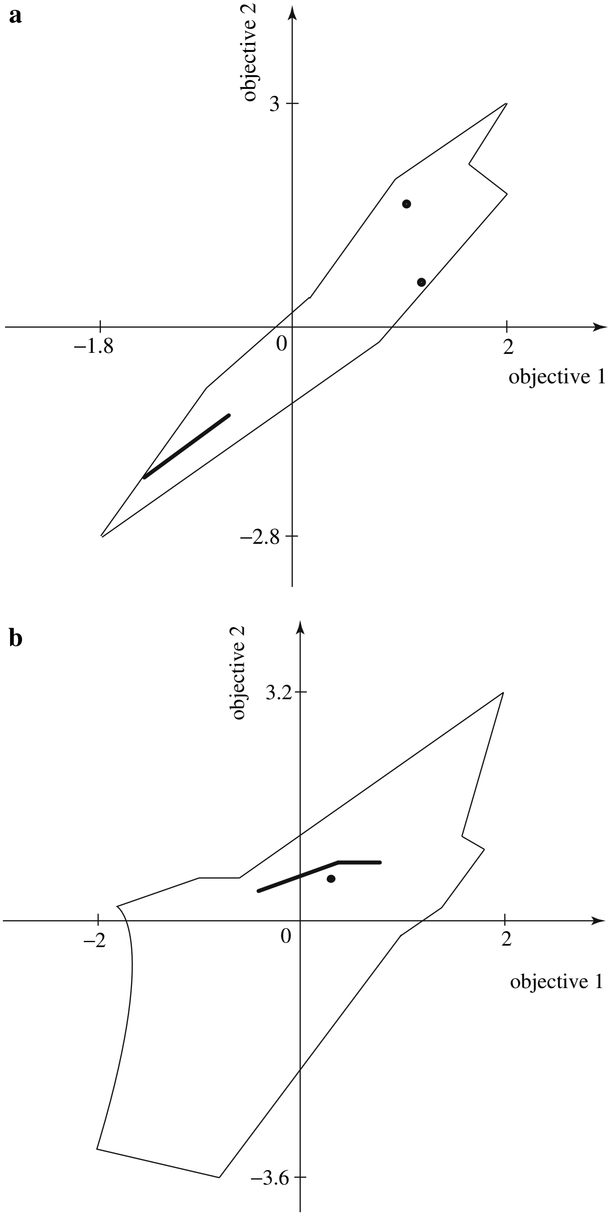

Fig. 2 Feasible areas of expected payoffs and payoff areas corresponding to the Pareto equilibrium solutions

equilibrium solutions of the game, and have also given a mathematical programming problem for Pareto equilibrium solutions. Furthermore, we have given a numerical example and have demonstrated that nondominated equilibrium solutions can be obtained by solving the formulated mathematical programming problem.

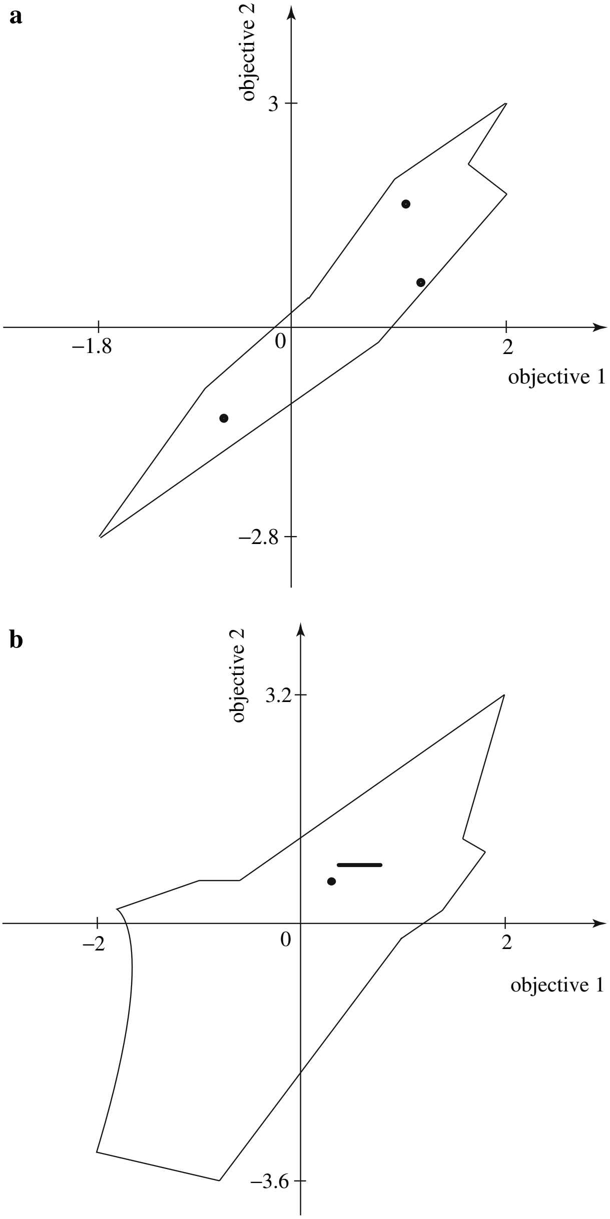

Fig. 3 Feasible areas of expected payoffs and payoff areas corresponding to the nondominated equilibrium solutions

References

Borm, P.E.M., Tijs, S.H., van den Aarssen, J.C.M.: Pareto equilibria in multiobjective games. In: Fuchsstein, B., Lengauer, T., Skaka, H.J. (eds.) Methods of Operations Research, pp. 303–312. Verlag Anton Hain Meisenheim GmbH, Frankfurt am Main (1988)

Borm, P., Vermeulen, D., Voorneveld, M.: The structure of the set of equilibria for two person multicriteria games. Eur. J. Oper. Res. **148**, 480–493 (2003)

Charnes, A., Huang, Z.M., Rousseau, J.J., Wei, Q.L.: Cone extremal solution of multi-payoff games with cross-constrained strategy set. Optimization **21**, 51–69 (1990)

Corley, H.W.: Games with vector payoffs. J. Optim. Theory Appl. **47**, 491–498 (1985)

Koller, D., Megiddo, N., von Stengel, B.: Efficient computation of equilibria for extensive two-person games. Games Econ. Behav. **14**, 247–259 (1996)

Krieger, T.: On Pareto equilibria in vector-valued extensive form games. Math. Meth. Oper. Res. **58**, 449–458 (2003)

Kuhn, H.W., Tucker, A.W.: Nonlinear programming. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Berkeley, California (1951)

Nishizaki, I., Notsu, T.: Nondominated equilibrium solutions of a multiobjective two-person nonzero-sum game and the corresponding mathematical programming problem. J. Optim. Theory Appl. **135**, (2007)

Nishizaki, I., Sakawa, M.: Equilibrium solutions for multiobjective bimatrix games incorporating fuzzy goals. J. Optim. Theory Appl. **86**, 433–458 (1995)

Nishizaki, I., Sakawa, M.: Equilibrium solutions in multiobjective bimatrix games with fuzzy payoffs and fuzzy goals. Fuzzy Sets Syst. **111**, 99–116 (2000)

Shapley, L.S.: Equilibrium points in games with vector payoffs. Nav. Res. Log. Q. **6**, 57–61 (1959)

Tamura, K., Miura, S.: Necessary and sufficient conditions for local and global nondominated solutions in decision problems with multi-objectives. J. Optim. Theory Appl. **28**, 501–523 (1979)

Von Stengel, B.: Efficient computation of behavior strategies. Games Econ. Behav. **14**, 220–246 (1996)

Voorneveld, M., Grahn, S., Dufwenberg, M.: Ideal equilibria in noncooperative multicriteria games. Math. Meth. Oper. Res. **52**, 65–77 (2000)

Voorneveld, M., Vermeulen, D., Borm, P.: Axiomatization of Pareto equilibria in multicriteria games. Games Econ. Behav. **28**, 146–154 (1999)

Wang, S.Y.: Existence of a Pareto equilibrium. J. Optim. Theory Appl. **79**, 373–384 (1993)

Wierzbicki, A.P.: Multiple criteria solutions in noncooperative game—theory part III, theoretical foundations. Kyoto Institute of Economic Research Discussion paper, No. 288 (1990)

Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)

Zeleny, M.: Games with multiple payoffs. Int. J. Game Theory **4**, 179–191 (1975)

Zhao, J.: The equilibria of a multiple objective game. Int. J. Game Theory **20**, 171–182 (1991)